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Modular differential equations and null vectors

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Abstract: We show that every modular differential equation of a rational conformal field theory comes from a null vector in the vacuum Verma module. We also comment on the implications of this result for the consistency of the extremal self-dual conformal field theories at $c = 24k$.

Keywords: Conformal Field Models in String Theory, AdS-CFT Correspondence.

Contents

1. Introduction

Every rational conformal field theory possesses a modular differential equation. This is to say, the different characters of the finitely many irreducible highest weight representations satisfy a common differential equation in the modular parameter. This fact was first observed, using the transformation properties of the characters under the modular group, in [1−4]; later developments of these ideas are described in [5−8]. Following the work of Zhu [9], the modular transformation properties of the characters were derived from first principles (see also [10]). Zhu's derivation suggests that the modular differential equation is a consequence of a null vector relation in the vacuum Verma module [11], see also [7]. In this paper we shall show that this idea is indeed correct.

The recent interest in this problem arose from the analysis of Witten concerning pure gravity in AdS_3 [12]. He suggested that the corresponding boundary theories should be holomorphically factorising bosonic conformal field theories at $c = 24k$ with $k = 1, 2, \ldots$, where $k \to \infty$ describes the classical limit of the AdS_3 theory. Furthermore, the corresponding chiral theories should be extremal, meaning that up to level $k + 1$ above the vacuum, the theory only consists of Virasoro descendants of the vacuum state. For $k = 1$, the resulting conformal field theory is the famous Monster theory [13, 14] (for a beautiful introduction see [15]), but for $k \geq 2$ an explicit realisation of these theories is so far not known. The above constraints, however, specify the character of these meromorphic conformal field theories uniquely [16, 12]. Furthermore, the $k = 2$ vacuum amplitudes are well-defined on higher genus Riemann surfaces [12], and the genus 2 amplitude of the $k = 3$ theory was shown to be consistent (by some other methods) [17]. Their method determines also the genus 2 partition functions uniquely up to $k \leq 10$. There has also been some evidence that suggests that these extremal theories account correctly for the corresponding gravity amplitudes [18−21].

On the other hand, it is not clear whether the theories with $k \geq 2$ do indeed exist. It was proposed in [11] that the analysis of their modular differential equations could shed light on this question. Since these theories are self-dual, they only have a single highest weight representation, and thus only a single character. One can then obtain an estimate for the order s of the differential equation that annihilates this character; this is proportional, for large k, to $s \sim \sqrt{k}$. On the other hand, if there is a direct relation between modular differential equations and null vectors in the vacuum Verma module, a modular differential equation at order s should imply that the vacuum Verma module has a null vector at conformal weight $2s$. This would then lead to a contradiction for $k \geq 42$ since the extremal theories do not have any null vectors at such low levels [11].

In [11] a more specific conjecture was made (and supported by some evidence). It was suggested that if a conformal field theory satisfies an order s modular differential equation, then $L^s_{-2}\Omega \in O_{[2]}$. (In particular, this conjecture implies the weaker statement that the vacuum Verma module has a null vector at level $2s$.) The original form of the conjecture has turned out to be incorrect: the example of Gaiotto [22] involving tensor products of the Monster theory demonstrates this fact. This example is however not in conflict with the weaker statement that the vacuum Verma module possesses a null vector at level $2s$ — albeit one that is of a somewhat different form. In fact, the tensor products of the Monster theory have many null vectors at levels that are even below the one suggested by the order of the modular differential equation!

In this paper we want to analyse the relation between modular differential equations and null vectors in detail. One of our main results is that every modular differential equation comes from a null vector in the vacuum Verma module (see (3.9)). We shall explain under which conditions this leads to a relation of the form $L_{-2}^s\Omega \in O_{[2]}$, thus giving in particular a null vector at level $2s$. We shall also explain in detail how the counterexample of Gaiotto avoids this conclusion; as we shall see, this is intimately related to the fact that the tensor product of two (or more) Monster theories has many other null vectors. We also comment on the fact that the existence of these additional null vectors can be seen from an analysis of the Monster theory character; the same is true for Witten's theory at $k = 2$, but, at least from the point of view of the character, there are no indications that the theories with $k \geq 3$ should have sufficiently many null vectors to avoid a contradiction along these lines.

The paper is organised as follows. In section 2, we explain in a comprehensive manner how the modular differential equation arises from the analysis of Zhu [9]. We also illustrate this with a simple and very explicit example, the Yang-Lee model at $c = -22/5$. In section 3 we show that a modular differential equation always leads to a null vector relation in the vacuum representation. We analyse under which conditions this implies that $L_{-2}^s\Omega \in O_{[2]}$, and how the counterexample of Gaiotto avoids this conclusion. Finally, we comment in section 4 on the implications of these considerations for the existence of the extremal self-dual conformal field theories at $k \geq 42$. In order to be comprehensive we have included a number of appendices: in appendix A we give a brief introduction to Zhu's algebra, while the torus recursion relations that underly the torus analysis of Zhu are derived (in a physicists manner) in appendix B. Appendix C describes our conventions for the Weierstrass functions and Eisenstein series, and details their modular properties, while appendix D describes one of the technical arguments of the paper.

2. The modular differential equation

Let us begin by explaining the structure of torus amplitudes in a rational conformal field theory. It is usually believed (and it follows in fact from the analysis of Zhu [9]) that the torus amplitudes can be described in terms of the characters of the highest weight representations of the conformal field theory. These characters satisfy a modular differential equation [3, 4] (for earlier work see [1, 2]). In this section we want to explain the origin of this differential equation from the point of view of Zhu [9].

Let V be a meromorphic conformal field theory (or vertex operator algebra). For each state $a \in V$ we have a vertex operator $V(a, z)$, whose modes we denote by a_n (using the usual physicists' conventions). The zero mode of a plays a special role, and we denote it by $o(a) \equiv a_0$. On the torus, it is more advantageous to use different coordinates; the associated modes are then denoted by $a_{[n]}$. All of this is explained in more detail in appendix A.

It follows from an elementary (but somewhat tedious) calculation due to Zhu (Proposition 4.3.5 of [9] — we sketch an outline of the argument in appendix B) that

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[-h_a]}b)\, q^{L_0}\Big) = \mathrm{Tr}_{\mathcal{H}}\Big(o(a)\, o(b)\, q^{L_0}\Big) + \sum_{k=1}^{\infty} G_{2k}(q)\, \mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[2k-h_a]}b)\, q^{L_0}\Big) . \quad (2.1)$$

Here the trace is taken in any highest weight representation $\mathcal{H}$ of the chiral algebra, and $G_n(q)$ denotes the n^{th} Eisenstein series; our conventions for the Eisenstein series (as well as their main properties) are summarised in appendix C. Next we apply (2.1) with a replaced by $L_{[-1]}a$, and use that $(L_{[-1]}a)_{[n]} = -(h_a+n)a_{[n]}$ (as follows from (A.4) upon taking a derivative), as well as $o(L_{[-1]}a) = (2\pi i)\, o(L_{-1}a + L_0 a) = 0$, which is a consequence of (A.7); this leads to (see Proposition 4.3.6 of [9])

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[-h_a-1]}b)\, q^{L_0}\Big) + \sum_{k\geq 1}(2k-1)G_{2k}(q)\ \mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[2k-h_a-1]}b)\, q^{L_0}\Big) = 0\ . \tag{2.2}$$

The term with $k=1$ does not contribute here since the trace of $o(a_{[-h_a+1]}b)$ vanishes — as follows from (B.11) in the appendix, it is a commutator and hence vanishes in the trace.

Equation (2.2) motivates now the following definition. Let $V[G_4(q), G_6(q)]$ be the space of polynomials in the Eisenstein series with coefficients in V. This is a module over the ring $R = \mathbb{C}[G_4(q), G_6(q)]$ which carries a natural grading given by the modular weight of each monomial; since G_4 and G_6 generate all modular forms, we have in particular that $G_{2k}(q) \in R$ for $k \geq 2$. Then we define $O_q(V)$ to be the submodule of $V[G_4(q), G_6(q)]$ generated by states of the form

$$O_q(V): \qquad a_{[-h_a-1]}b + \sum_{k\geq 2}(2k-1)\, G_{2k}(q)\, a_{[2k-h_a-1]}\, b\,, \tag{2.3}$$

where $a, b \in V$. Here the sum is finite, as $a_{[n]}$ annihilates b for sufficiently large n. By (2.2), it is now clear that

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(v)\, q^{L_0}\Big) = 0 \qquad \text{if } v \in O_q(V)\ . \tag{2.4}$$

This is true for every character, i.e. independent of the highest weight representation $\mathcal{H}$ that is being considered. For later convenience we also note that

$$a_{[-h_a-n]}b - (-1)^n \sum_{2k\geq n+1} \binom{2k-1}{n} G_{2k}(q)\, a_{[2k-h_a-n]}b \in O_q(V)\,, \quad \forall n \geq 1\,, \tag{2.5}$$

as can be seen by evaluating the above identity repeatedly with a being replaced by $L_{[-1]}a$.

Suppose now that for a conformal field theory we can find an integer s and modular forms $g_r(q)$ of weight $2(s-r)$ such that[1]

$$\big(L_{[-2]}\big)^s\Omega + \sum_{r=0}^{s-2} g_r(q)\, \big(L_{[-2]}\big)^r\Omega \in O_q(V)\ . \tag{2.6}$$

We then claim that all the characters $\chi_{\mathcal{H}}(q) := \mathrm{Tr}_{\mathcal{H}}\big(q^{L_0-\frac{c}{24}}\big)$ of the conformal field theory satisfy a common modular covariant differential equation, i.e. an equation of the form

$$\left[D^s + \sum_{r=0}^{s-2} f_r(q) D^r\right]\chi_M(q) = 0\ . \tag{2.7}$$

[1] As is explained in [9], the existence of such a vector follows for example from the C_2 condition that is believed to hold for every rational conformal field theory — see also appendix A.2.

Here D^s is the order s differential operator (see appendix C)

$$D^s = D_{2s-2}\, D_{2s-4} \cdots D_2\, D_0\,, \qquad \text{with} \qquad D_r = q\frac{d}{dq} - \frac{r}{4\pi^2} G_2(q) = q\frac{d}{dq} - \frac{r}{12} E_2(q)\,, \tag{2.8}$$

and $f_r(q)$ is a modular form of weight $2(s-r)$.[2]

To show this, note that because of the defining property of $O_q(V)$ (2.4), we know that the character of the zero mode of the left hand side of (2.6) vanishes. On the other hand, each term in this expression can be turned into a differential operator

$$\mathrm{Tr}_{\mathcal{H}}\Big(o\big((L_{[-2]})^r \Omega\big)\, q^{L_0 - \frac{c}{24}}\Big) = P_r(D)\, \mathrm{Tr}_{\mathcal{H}}\Big(q^{L_0 - \frac{c}{24}}\Big)\,, \tag{2.9}$$

where $P_r(D)$ is a modular covariant differential operator of order r with modular weight $2r$. To see (2.9) we note that for $r=1$ we obtain directly

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(L_{[-2]}\Omega) q^{L_0 - \frac{c}{24}}\Big) = (2\pi i)^2\, \mathrm{Tr}_{\mathcal{H}}\Big(\Big(L_0 - \frac{c}{24}\Big) q^{L_0 - \frac{c}{24}}\Big) = (2\pi i)^2 \Big(q\frac{d}{dq}\Big)\, \mathrm{Tr}_{\mathcal{H}}\Big(q^{L_0 - \frac{c}{24}}\Big)\,, \tag{2.10}$$

which is modular covariant since the character has modular weight 0. The case of general r follows by applying (2.1) (which clearly still works if we replace q^{L_0} by $q^{L_0 - c/24}$)

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}}\Big(o(L_{[-2]}(L_{[-2]})^r\Omega)\, q^{L_0 - \frac{c}{24}}\Big) \;=\; & (2\pi i)^2 q\frac{d}{dq}\, \mathrm{Tr}_{\mathcal{H}}\Big(o((L_{[-2]})^r\Omega)\, q^{L_0 - \frac{c}{24}}\Big) \\ & + 2r G_2(q)\; \mathrm{Tr}_{\mathcal{H}}\Big(o((L_{[-2]})^r\Omega)\, q^{L_0 - \frac{c}{24}}\Big) \\ & + \sum_{k\geq 2} G_{2k}(q)\; \mathrm{Tr}_{\mathcal{H}}\Big(o(L_{[2k-2]}(L_{[-2]})^r\Omega)\, q^{L_0 - \frac{c}{24}}\Big)\,. \end{aligned} \tag{2.11}$$

In the last line we commute the positive $L_{[2k-2]}$ modes to the right, using the Virasoro commutation relations. The final result is a vector of the form $(L_{[-2]})^{r+1-k}\Omega$, which leads to a differential operator of lower order, multiplied by the modular form of appropriate weight. The first two terms, on the other hand, just produce the covariant derivative D_{2r} for a form of weight $2r$. Collecting all terms, we get the desired operator $P_r(D)$. Note that the leading term of $P_r(D)$ is proportional to D^r; for the first few values of r, the explicit formula for $P_r(D)$ is given in appendix B.1. This completes the derivation of the modular differential equation.

2.1 A simple example

Let us illustrate this construction with a simple example, the Yang-Lee minimal model at $c = -\frac{22}{5}$. This is the 'simplest' minimal model since it only has two highest weight representations, the vacuum representation at $h=0$ as well as the representation at $h = -\frac{1}{5}$. The vacuum representation has a null vector at level 4,

$$\mathcal{N} = \Big(L_{[-4]} - \frac{5}{3} L_{[-2]}^2\Big)\,\Omega\,. \tag{2.12}$$

[2]We shall use two different conventions for the Eisenstein series, namely $G_n(q)$ and $E_n(q)$, in this paper; the two functions only differ by an overall normalisation constant, see appendix C.

We want to use $\mathcal{N}$ to obtain an expression of the form (2.6). To this end we observe that $L_{[-4]}\Omega$ is already in $O_q(V)$ since (2.5) implies that

$$O_q(V) \ni L_{[-4]}\Omega - \sum_{k\geq 2}\binom{2k-1}{2} G_{2k}(q)\, L_{[2k-4]}\Omega = L_{[-4]}\Omega\ . \tag{2.13}$$

Since $\mathcal{N}$ is a null vector, the sought-after relation is then simply

$$L_{[-2]}^2\Omega \in O_q(V)\ . \tag{2.14}$$

Using the explicit expression for (2.9) derived in appendix B.1, we obtain the differential equation

$$0 = \operatorname{Tr}_{\mathcal{H}}\Big(o(L_{[-2]}L_{[-2]}\Omega)\, q^{L_0-\frac{c}{24}}\Big) = (2\pi i)^4 \left[D^2 - \frac{11}{3600}\, E_4(q)\right] \chi_{\mathcal{H}}(q)\ . \tag{2.15}$$

The two characters of the Yang-Lee model are explicitly given as (see for example [23])

$$\chi_0(q) = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}\left(q^{\frac{(20n-3)^2}{40}} - q^{\frac{(20n+7)^2}{40}}\right) \tag{2.16}$$

$$\chi_{-1/5}(q) = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}\left(q^{\frac{(20n-1)^2}{40}} - q^{\frac{(20n+9)^2}{40}}\right)\ , \tag{2.17}$$

where $\eta(q)$ is the usual Dedekind eta function

$$\eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\ . \tag{2.18}$$

One easily checks (using for example Mathematica) that the two characters are indeed the two solutions of this second order differential equation. We have also performed the analogous analysis for the Ising model.

2.2 Relation to the null vector

In the above example, the vector of the form (2.6) in $O_q(V)$ was a direct consequence of a null vector relation in the vacuum representation, see (2.12). This is actually generally true: a vector of the form (2.6) in $O_q(V)$ can only exist if the vacuum representation has a null vector at level $2s$. To see this we recall that $V[G_4(q), G_6(q)]$ carries two grades: the *conformal weight* of the vectors in V (with respect to $L_{[0]}$), and the *modular weight* of the coefficient functions (that are polynomials in G_4 and G_6). Furthermore, the relations that define $O_q(V)$ are homogeneous with respect to the grade that is the sum of these two grades, as is manifest from (2.3).

Since the relation (2.6) is a relation in $V[G_4(q), G_6(q)]$ it must hold separately for every conformal weight and every modular weight. If we consider the component at conformal weight $2s$ and modular weight zero, we therefore get a relation of the form

$$\big(L_{[-2]}\big)^s\Omega + \sum_j a^j_{[-h(a^j)-1]}\, b^j = 0\,, \tag{2.19}$$

where $h(a^j)$ is the conformal weight (with respect to $L_{[0]}$) of a^j. This is necessarily a non-trivial relation in the Verma module since $L_{[-2]}$ is not of the form $a_{[-h_a-1]}$ for any a. Such a non-trivial relation is usually called a null vector. We mention in passing that it implies that $\left(L_{[-2]}\right)^s\Omega$ vanishes in the C_2 quotient space of Zhu (that is briefly discussed in appendix A.2), as was already mentioned in [11].

3. Reconstructing the null vector

As we have seen above, a vector of the form (2.6) in $O_q(V)$ implies that the characters of the theory satisfy a common order s modular differential equation. We have also shown that such a relation in $O_q(V)$ can only exist if the vacuum representation has a null vector at conformal weight $2s$, see (2.19).

We would now like to show a partial converse to these statements, namely that every modular differential equation implies that the vacuum Verma module has a null vector. We shall assume that Zhu's algebra is semisimple, as is known to be the case for rational conformal field theories (in the mathematical sense) [9]. In particular, this is the case for the self-dual theories, for which Zhu's algebra is one-dimensional, consisting only of the identity. In this section we only sketch the idea of the proof; more details of the calculation can be found in appendix D .

3.1 The underlying vector

Suppose now that we have a modular covariant differential equation of the form (2.7) that annihilates all characters of the conformal field theory. Using the arguments of section 2 in reverse order, it is easy to see that there is then a vector $K(q)$ of the form

$$K(q) \equiv \left(L_{[-2]}\right)^s\Omega + \sum_{r=0}^{s-2} g_r(q)\left(L_{[-2]}\right)^r\Omega \tag{3.1}$$

where each $g_r(q)$ is a modular form of weight $2(s-r)$, that has the property that

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(K(q))\, q^{L_0-\frac{c}{24}}\Big) = 0 \tag{3.2}$$

for all characters of the conformal field theory. Let us consider the limit

$$\lim_{q\to 0} q^{\frac{c}{24}-h}\,\mathrm{Tr}_{\mathcal{H}}\Big(o(K(q))\, q^{L_0-\frac{c}{24}}\Big) = 0\,, \tag{3.3}$$

where h is the conformal weight of the highest weight state in $\mathcal{H}$. In this limit only the highest weight states $\mathcal{H}^0$ in $\mathcal{H}$ contribute, and we conclude that

$$\mathrm{Tr}_{\mathcal{H}^0}\Big(o(K(0))\Big) = 0\ . \tag{3.4}$$

3.2 Using Zhu's theorem

The above argument has shown that $K(0)$ acts trivially in the trace of an arbitrary highest weight representation. The action of the elements of V on highest weight states is captured

by Zhu's algebra (for a brief introduction see appendix A.1). If Zhu's algebra is semisimple (as we shall assume) then the fact that $K(0)$ is trivial in all traces implies that $K(0)$ must equal a commutator in Zhu's algebra; this is shown in appendix D. This implies that up to commutator terms of the form $d^l_{[-h(d^l)+1]}\, e^l$, $K(0)$ lies in $O_{[1,1]}$, the subspace by which we quotient to obtain Zhu's algebra $A(V)$. On the other hand, $O_{[1,1]}$ is closely related to $O_q(V)$, since again up to a commutator terms, every element in $O_{[1,1]}$ can be obtained as the limit $q \to 0$ of an element $H_j(q) \in O_q(V)$ (see again appendix D).
Taking these statements together they now imply that $K(0)$ can be written as

$$K(0) - \sum_l d^l_{[-h(d^l)+1]}\, e^l - \sum_j H^j(0) = 0 \ , \tag{3.5}$$

where each $H^j(q)$ is an element of $O_q(V)$

$$H^j(q) = a^j_{[-h(a^j)-1]} b^j + \sum_{k\geq 2} (2k-1) G_{2k}(q)\, a^j_{[2k-h(a^j)-1]} b^j \in O_q(V) \tag{3.6}$$

for some suitable set of a^j and b^j. Next we define an element $N(q) \in V[G_4, G_6]$ by

$$N(q) \equiv K(q) - \sum_l d^l_{[-h(d^l)+1]}\, e^l - \sum_j H^j(q) \ . \tag{3.7}$$

By construction, $N(0) = 0$, and hence $N(q)$ is proportional to q. We can then divide by q, and repeat the above argument. Recursively this allows us to prove that

$$K(q) = \sum_l f_l(q)\, d^l_{[-h(d^l)+1]}\, e^l + \sum_j h_j(q)\, H^j(q) \tag{3.8}$$

in $V[q]$, where $V[q]$ consists of vectors in V with coefficients that are formal power series in q. If we assume that the theory is C_2-finite (as is expected to be the case for any rational theory) one can show that only finitely many terms appear and that the power series have a non-trivial radius of convergence; this is explained in appendix D.

Putting everything together, we can now use (3.8) as well as (3.1) and (3.6) to arrive at the identity

$$\begin{aligned} &(L_{[-2]})^s \Omega + \sum_{i=0}^{s-1} g_i(q) (L_{[-2]})^i \Omega \\ &\quad = \sum_l f_l(q) d^l_{[-h(d^l)+1]} e^l + \sum_j h_j(q) \Big(a^j_{[-h(a^j)-1]} b^j + \sum_{k\geq 2} (2k-1) G_{2k}(q)\, a^j_{[2k-h(a^j)-1]} b^j \Big) \end{aligned} \tag{3.9}$$

as a relation in $V[q]$. This defines the sought after 'null vector' relation in the vacuum Verma module. Obviously, the full expression is not homogeneous with respect to conformal weight, and therefore each component (i.e. the terms of each fixed conformal weight) must vanish separately (and indeed for any power of q). Some of these relations may be trivial in the Verma module, but not all of them can if the original modular differential equation from which we started was non-trivial.

3.3 Consequences

We have thus shown that every modular differential equation comes from a null vector in the vacuum Verma module. We would now like to obtain more detailed information from (3.9). For the application to the extremal self-dual conformal field theories, it is for instance also important to determine the conformal weights of the constituent null vectors. In particular, one may expect that the term of highest conformal weight on the left-hand-side — this is the vector $(L_{-2})^s\Omega$ — should be part of a non-trivial null vector relation.

In order to motivate this proposal we observe that the coefficients of the vectors of the left-hand-side of (3.9) are all analytic functions in q on the unit disc, $|q| < 1$. Therefore the same has to be true for the coefficients on the right-hand-side. Generically, one should then expect that the functions $f_l(q)$ and $h_j(q)$ will also be analytic functions on $|q| < 1$; as we shall discuss later on, there are however situations where this is not the case.

Now we recall that $V[G_4(q), G_6(q)]$ has two gradings, namely the ones given by conformal weight and modular weight. By construction $(L_{[-2]})^s\Omega$ has modular weight 0 and conformal weight $2s$. If $f_l(q)$ and $h_j(q)$ are indeed analytic, then the only terms of modular weight 0 on the right hand side of (3.9) have constant coefficients. Moreover, comparing the conformal weights, only terms of $L_{[0]}$-weight $2s$ can contribute. Thus we can conclude that we have an identity of the form

$$(L_{[-2]})^s\Omega = \sum_j{}' a^j_{[-h(a^j)-1]} b^j + \sum_l{}' d^l_{[-h(d^l)+1]} e^l \,, \tag{3.10}$$

where the prime over the sum indicates that we only include states of $L_{[0]}$-weight $2s$, i.e. terms with $h(a^j) + h(b^j) + 1 = 2s$ and $h(d^l) + h(e^l) - 1 = 2s$. Because of the 'commutator terms', i.e. the first sum in (3.10), this identity does not quite imply that $L^s_{[-2]}\Omega \in O_{[2]}$. However, for the case of the extremal self-dual theories at $c = 24k$ we can show (see section 4 below) that this is so, and hence that (3.10) defines indeed a non-trivial null vector relation.

In the above argument we have used that there are no holomorphic functions of negative modular weight; in particular, this implied that $h_j(q)G_{2k}(q)$ had modular weight greater or equal to $2k$, and hence could not contribute to the identity (3.10). However, as soon as we allow h_j to be meromorphic, we can no longer guarantee this. For example, we can then construct other contributions to (3.10) from terms with $k \neq 0$ by choosing $h_j(q) = G_{2k}(q)^{-1}$. We will now discuss an example of such a situation.

3.4 A counterexample

It was observed in [22] that for the tensor product of two (or more) Monster theories, there exist modular differential equations that do not come from relations of the type (3.10). As we shall explain in the following, this 'counterexample' to (3.10) can be traced back to the failure of h_j to be holomorphic. We shall also see that this is only compatible with the holomorphicity of (3.9) because the Monster theory (and indeed the tensor products of the Monster theory) has many other null vectors at low levels. These null vectors are necessary to guarantee that the apparent non-holomorphic terms on the right-hand-side of (3.9) in fact vanish in the vacuum representation. Thus it seems that (3.10) can only be avoided if the theory has other non-trivial null vectors at low levels.

3.4.1 The Monster theory

To set up the notation we first recall a few facts about the case of a single Monster theory; for an introduction to these matters see for example [15]. The Monster theory has no fields of conformal dimension one, and 196884 fields of conformal dimension two. The latter consist of the stress-energy tensor whose modes L_n satisfy a Virasoro algebra at central charge $c = 24$,

$$[L_m, L_n] = (m-n)L_{m+n} + 2m(m^2-1)\delta_{m,-n} \; . \tag{3.11}$$

The remaining 196883 fields W^i transform in an irreducible representation of the Monster group and satisfy the commutation relations

$$\begin{aligned} [L_m, W^i_n] &= (m-n)W^i_{m+n} \\ [W^i_m, W^j_n] &= \frac{1}{6}\delta^{ij} m(m^2-1)\delta_{m,-n} + \frac{1}{12}\delta^{ij}(m-n)L_{m+n} \\ &\quad + h^{ij}_k (m-n) W^k_{m+n} + f^{ij}_\alpha V^\alpha_{m+n} \, , \end{aligned} \tag{3.12}$$

where V^α_l are the modes of the primary fields at conformal weight three that transform in the 21296876-dimensional irreducible representation of the Monster group. The coefficients h^{ij}_k are totally symmetric in all three indices, and define the structure constants of the so-called Griess algebra. In our conventions, the metric on the space of the W^i fields is orthonormal, so we can raise and lower the i, j, k indices freely.

The Monster conformal field theory has many non-trivial relations; the first non-trivial relation already occurs at level four since we have the identity (see for example [25])[3]

$$\mathcal{N}_4 = L^2_{-2}\Omega + \frac{36}{11}L_{-4}\Omega - \frac{12}{30503}\sum_i W^i_{-2}W^i_{-2}\Omega = 0 \; . \tag{3.13}$$

This null-relation does, however, not directly lead to a differential equation since it is not of the form (3.10). As was already explained in [11], the character of the Monster theory $\chi_M(q)$ satisfies only a third order differential equation

$$\left[D^3 + \frac{16}{31}E_6(q) - \frac{290}{279}E_4(q)\, D\right]\chi_M(q) = 0 \; . \tag{3.14}$$

This differential equation can be obtained from the null vector at level six (see again [25])

$$\mathcal{N}_6 = L^3_{-2}\Omega + \frac{41}{8}L^2_{-3}\Omega + \frac{15623}{1488}L_{-4}L_{-2}\Omega + \frac{873}{31}L_{-6}\Omega - \frac{1}{124}\sum_i W^i_{-4}W^i_{-2}\Omega = 0 \; . \tag{3.15}$$

In fact, it is easy to see that evaluating the trace of $V_0(\mathcal{N}_6)$ as in section 2 (where in the definition of $\mathcal{N}_6$ we replace the L_{-n} modes by $L_{[-n]}$ modes, and similarly for the W^i_{-n})

[3] This follows from the equation after (2.9) in [25] upon rewriting his modes x^i with $i = 1, \ldots, 196884$ in terms of the W^i and L.

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leads to the above modular differential equation. (In order to do this calculation, one also needs to use the commutation relations of the W^i-modes.)

There is an independent null vector at level eight, which is of the form[4]

$$\begin{aligned}\mathcal{N}_8 = h_{ijk}W^i_{-4}W^j_{-2}W^k_{-2}\Omega - H_0\Big[&\frac{503352}{8072203}L_{-8}\Omega + \frac{81048}{8072203}L_{-6}L_{-2}\Omega \\ &+\frac{34565}{8072203}L_{-5}L_{-3}\Omega + \frac{26403}{16144406}L_{-4}L_{-4}\Omega + \frac{110221}{96866436}L_{-4}L_{-2}L_{-2}\Omega \\ &+\frac{3193}{16144406}L_{-3}L_{-3}L_{-2}\Omega + \frac{5210}{24216609}L_{-2}L_{-2}L_{-2}L_{-2}\Omega\Big]\,, \end{aligned} \tag{3.16}$$

where $H_0 = h_{ijk}h^{ijk}$, which equals in our conventions $H_0 = 196883\,\frac{6929}{6} = \frac{1364202307}{6}$. By the same token as above (and with somewhat more effort — in particular, we now also have to use the null vector $\mathcal{N}_4$ in order to express the term $h_{ijk}W^i_{[0]}W^j_{[-2]}W^k_{[-2]}\Omega$ that appears in the course of this calculation in terms of Virasoro generators) it leads to the fourth order modular differential equation

$$\left[D^4 - \frac{73421}{93780}E_4(q)\,D^2 + \frac{527029}{562680}E_6(q)\,D - \frac{1259}{2605}E_4^2(q)\right]\chi_M(q) = 0\,. \tag{3.17}$$

This differential equation is actually linearly independent from the other fourth order modular differential equation of the Monster theory, namely the one coming from the null vector $L_{-2}\mathcal{N}_6$. The latter differential equation equals

$$\left[D^4 - \frac{290}{279}E_4(q)\,D^2 + \frac{722}{837}E_6(q)\,D - \frac{8}{31}E_4^2(q)\right]\chi_M(q) = 0\,, \tag{3.18}$$

which is in fact simply equal to the D-derivative of (3.14). Taking the difference of (3.17) and (3.18) the Monster theory therefore also satisfies a modular differential equation of order two,

$$\left[E_4(q)D^2 + \frac{71}{246}E_6(q)\,D - \frac{36}{41}E_4^2(q)\right]\chi_M(q) = 0\,. \tag{3.19}$$

[Another way of saying this, is that this is the modular differential equation that comes from the null vector

$$\mathcal{M}_8 = (2\pi i)^{-8}\frac{16151}{2982996}\left(\frac{24216609}{5210\,H_0}\mathcal{N}_8 - L_{-2}\mathcal{N}_6\right)\,.\Big] \tag{3.20}$$

Note that the existence of this second order modular differential equation is not in conflict with what was said above (or in [11]), since (3.19) is not holomorphic in the above sense: if we divide by $E_4(q)$ to obtain a differential equation whose leading term is D^2, the coefficient of the term proportional to D is not holomorphic but only meromorphic. If we allow for meromorphic coefficients, every self-dual conformal field theory obviously also satisfies a first order modular differential equation (see also [8]).

[4]Such a null vector must exist since, up to level 10, all states that are Monster invariant can be expressed in terms of Virasoro descendants of the vacuum. The coefficients can then be fixed by evaluating the inner products with all Virasoro descendants.

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3.4.2 Tensor products of Monster theories

Now let us turn to the case of the tensor product of two Monster theories. (As we shall see momentarily, the answer for the tensor product of an arbitrary number of Monster theories can be understood once we have done so for the two-fold tensor product.) It is not difficult to show that if (3.10) was true, an order s modular differential equation for the tensor product of the two Monster theories would imply that

$$\left(L^{(1)}_{-2} + L^{(2)}_{-2}\right)^s \Omega \in O_{[2]} \ . \tag{3.21}$$

Given the arguments of [11, 22] it is easy to see that (3.21) can only hold for $s \geq 5$. On the other hand, one finds that the tensor product of two Monster theories actually satisfies a fourth order differential equation [22], namely

$$\left[D^4 - \frac{175117}{45756} E_4(q)\, D^2 + \frac{47539165}{11255976}\, E_6(q)\, D - \frac{12838}{52111}\, E_4^2(q)\right] \chi_M^2(q) = 0 \ . \tag{3.22}$$

We now want to explain how to obtain this differential equation from a null vector in the vacuum Verma module. First we observe that the leading term D^4 in (3.22) comes from the vector

$$\begin{aligned}\left(L^{(1)}_{[-2]} + L^{(2)}_{[-2]}\right)^4 \Omega = \Bigg[&\left(L^{(1)}_{[-2]}\right)^4 + 4\left(L^{(1)}_{[-2]}\right)^3 L^{(2)}_{[-2]} \\ &+6\left(L^{(1)}_{[-2]}\right)^2\left(L^{(2)}_{[-2]}\right)^2 + 4L^{(1)}_{[-2]}\left(L^{(2)}_{[-2]}\right)^3 + \left(L^{(2)}_{[-2]}\right)^4\Bigg]\Omega \ .\end{aligned} \tag{3.23}$$

In the following we want to show how this vector can be expressed, up to terms of lower conformal weight, in terms of elements in $O_q(V)$. The terms in $O_q(V)$ vanish inside any trace, and the terms of lower conformal weight can be expressed in terms of Virasoro generators, and hence give rise to the lower coefficients in (3.22).[5]

The various terms in (3.23) can now be rewritten as follows. First of all, we observe that every element in $O_q(V)$ is of the form

$$O_q(V): \quad v + \sum_{n\geq 2} G_n(q) v_n \,, \qquad \text{where} \quad v \in O_{[2]} \,, \tag{3.24}$$

and that for any $v \in O_{[2]}$, there is such an element in $O_q(V)$. We call v the 'head', and the remaining terms the 'tail'. Note that the conformal weights of the terms in the tail are always strictly smaller than that of v.

Now we can use the null vector $\mathcal{N}_8$ (or $L_{[-2]}\mathcal{N}_6$) to express $(L^{(i)}_{[-2]})^4\Omega$, where $i = 1, 2$, in terms of a vector in $O_{[2]}$. This can be taken to form the head of an element in $O_q(V)$, and hence we can rewrite $(L^{(i)}_{[-2]})^4\Omega$, up to elements of lower conformal weight that come from the tail, as an element of $O_q(V)$. Similarly, we can reduce $(L^{(1)}_{[-2]})^3 L^{(2)}_{[-2]}\Omega$ by using the null

[5] Strictly speaking we also have to guarantee that the resulting terms of lower conformal weight can be expressed in terms of powers of $(L^{(1)}_{[-2]} + L^{(2)}_{[-2]})$, but this can indeed be arranged — this is again a consequence of the fact that there are two independent null vectors at level eight.

vector $\mathcal{N}_6^{(1)} \otimes L_{[-2]}^{(2)}\Omega$, and likewise for the term $L_{[-2]}^{(1)}(L_{[-2]}^{(2)})^3\Omega$. The only difficult term is $(L_{[-2]}^{(1)})^2\,(L_{[-2]}^{(2)})^2\Omega$ for which this is not possible — in fact, this is the reason why (3.21) with $s = 4$ does not hold. We now want to explain how this can be circumvented by making use of the null vector $\mathcal{M}_8$.

As we have seen above, the single Monster theory has a null vector at level 8, $\mathcal{M}_8$, that lies entirely inside $O_{[2]}$, $\mathcal{M}_8 \in O_{[2]}$. Let us denote by $O_{\mathcal{M}_8}$ its tail, so that $\mathcal{M}_8 + O_{\mathcal{M}_8} \cong O_{\mathcal{M}_8} \in O_q(V)$; this is explicitly given (up to an overall normalisation) as

$$O_{\mathcal{M}_8} = \left[G_4(q)L_{[-2]}^2 - \frac{497}{41}G_6(q)L_{[-2]} - \frac{26412}{41}G_4(q)^2\right]\Omega\,, \tag{3.25}$$

where we have made use of the null vector $\mathcal{N}_4$ at level four to rewrite the term $W_{[-2]}^iW_{[-2]}^i\Omega$ that appeared in the course of this calculation in terms of $L_{[-2]}^2\Omega$.

The same argument also applies to the null vector $\mathcal{M}_{10} := L_{[-2]}\mathcal{M}_8$. Up to an overall constant, its tail is

$$O_{\mathcal{M}_{10}} = \Big(G_4(q)L_{[-2]}^3 + \lambda_1 G_6(q)L_{[-2]}^2 + \lambda_2 G_4^2(q)L_{[-2]} + \lambda_3 G_4(q)G_6(q)\Big)\Omega\,, \tag{3.26}$$

where

$$\lambda_1 = \frac{2334255}{1158254}\,,\qquad \lambda_2 = -\frac{451255338}{579127}\,,\qquad \lambda_3 = -\frac{10493019690}{579127}\,, \tag{3.27}$$

and we have used the null vector relation $\widehat{\mathcal{N}}_6 = 0$ with

$$\begin{aligned}\widehat{\mathcal{N}}_6 \,=\, h_{ijk}W_{-2}^iW_{-2}^jW_{-2}^k\Omega - H_0\bigg[&\frac{20403}{196883}\,L_{-6}\Omega + \frac{5607}{393766}\,L_{-4}L_{-2}\Omega \\ &+\frac{279}{196883}\,L_{-3}L_{-3}\Omega + \frac{8837}{2362596}\,L_{-2}L_{-2}L_{-2}\Omega\bigg]\,.\end{aligned} \tag{3.28}$$

Now we can combine these null vectors to write

$$\begin{aligned}&\left[\left(L_{[-2]}^{(1)}\right)^2\left(L_{[-2]}^{(2)}\right)^2 + \frac{497}{41\,\lambda_1}\left(L_{[-2]}^{(1)}\right)^3\left(L_{[-2]}^{(2)}\right)\right]\Omega + \text{terms of lower conformal weight} \\ &\quad = \frac{1}{G_4(q)}\left\{\left(L_{[-2]}^{(1)}\right)^2\Omega^{(1)} \otimes O_{\mathcal{M}_8}^{(2)} + \frac{497}{41\,\lambda_1}\,O_{\mathcal{M}_{10}}^{(1)} \otimes \left(L_{[-2]}^{(2)}\right)\Omega^{(2)}\right\} \in O(q)\,.\end{aligned} \tag{3.29}$$

Generically, such an identity will involve coefficients that are not holomorphic in q, since the terms in the bracket on the right-hand-side will not automatically be divisible by $G_4(q)$. However, for the specific linear combination that we have chosen — i.e. for the relative coefficient $\frac{497}{41\lambda_1}$ — the expression is actually holomorphic. To see this we observe that the coefficients that appear in the bracket are proportional to Eisenstein series G_n with $n = 6, 8, 10$. Except for G_6, these Eisenstein series are automatically divisible by G_4. Thus we only need to guarantee that the coefficient of G_6 vanishes, and this is precisely achieved by the above linear combination.

Translating this analysis back into the language of section 3.2, it is now clear that the identity (3.9) that corresponds to the fourth order modular differential equation (3.22) is of the form

$$\left(L_{-2}^{(1)} + L_{-2}^{(2)}\right)^4\Omega + \sum_{i=0}^{3} g_i(q)\left(L_{-2}^{(1)} + L_{-2}^{(2)}\right)^i\Omega = \frac{1}{G_4(q)}H(q) + \sum_l \hat{h}_l(q)H^l(q)\,, \tag{3.30}$$

where $H(q)$ is the element of $O_q(V)$ defined by the curly bracket in (3.29), and the other $\hat{h}_l(q)$ are holomorphic. In this case the functions $h_j(q)$ of (3.9) involve thus one meromorphic (but not holomorphic) function, namely $1/G_4(q)$.

It should be clear from this analysis that such a non-holomorphic coefficient function h_j in (3.9) can only appear if the theory has sufficiently many null vectors to guarantee that all non-holomorphic terms on the right-hand-side of (3.9) are actually zero. (In the above case we had to use, for both theories, the null vector at level four, the two null vectors at level six, and the null vector at level eight.) For larger conformal weight the situation becomes even more constraining since then the tail will generically also involve Eisenstein series G_n with $n > 14$, none of which are divisible by G_4. Thus there will be even more coefficients that will need to be cancelled!

Finally, let us comment on the question of how this analysis generalises to higher tensor powers of the Monster theory. It is clear from the above analysis that for the k-fold tensor product we can always construct a modular differential equation of order $k+2$. To see this we expand out

$$\left(\sum_{i=1}^{k} L_{-2}^{(i)}\right)^{k+2} \Omega \, . \tag{3.31}$$

Then each term will either be proportional to $(L_{-2}^{(i)})^3\Omega$ for some i — such terms lie in $O_{[2]}$ by virtue of the null vector $\mathcal{N}_6$ — or to terms of the form $(L_{-2}^{(i)})^2(L_{-2}^{(j)})^2\Omega$ which can be dealt with as explained above. Thus using the above methods we can construct a modular differential equation at order $k+2$. On the other hand, this seems to be the minimal order for which such a differential equation exists [22]. Thus there do not seem to be any additional cancellations beyond what is already visible for the case of the tensor product of two Monster theories. Finally, we should stress that the k-fold tensor product has a plethora of low-lying null vectors: there are at least k linearly independent null vectors at level 4, $2k$ at level 6, k additional ones at level 8, *etc*, that are relevant for this analysis.

4. Application to extremal self-dual CFTs

In this final section we want to comment on the implications of these considerations for the existence of the extremal self-dual conformal field theories at $c = 24k$ that were proposed by Witten [12]. As was shown in [11], these theories satisfy a modular differential equation of degree s where, for large k, $s \sim \sqrt{k}$.

As we have shown in section 3 above, every modular differential equation comes from a null vector in the vacuum Verma module, see (3.9). Provided that f_l and h_j are holomorphic for $|q| < 1$, the null vector relation (3.9) implies that (3.10) holds. We now want to show that (3.10) leads to a contradiction for $k \geq 42$. Thus the extremal conformal field theories can only be consistent for large k, provided that the assumption about the analyticity of f_l and h_j is not satisfied; we shall comment on this possibility further below.

Suppose then the extremal conformal field theories have a 'null vector-relation' of the form (3.10) at conformal weight $2s$. For $k \geq 42$ this relation is at $L_{[0]}$-weight $2s \leq k$, and thus arises at a weight where the proposed conformal field theory only possesses Virasoro descendants of the vacuum. This then leads to a contradiction: by the above argument, the right hand side can only contain Virasoro operators, which we may bring to the standard Poincaré-Birkhoff-Witt basis. We now claim that no term $(L_{[-2]})^s\Omega$ can arise in the process. Consider first the terms $a_{[-h(a)-1]}b$. Since b can only be a Virasoro descendant of the vacuum, we can write it as a sum of terms

$$L_{[-n_1]}\cdots L_{[-n_N]}\Omega\,, \tag{4.1}$$

where all $n_l \geq 2$. Since the level of b is $h(b)$, we have necessarily that $N \leq \lfloor\frac{h(b)}{2}\rfloor$, where $\lfloor\cdot\rfloor$ denotes the truncated part. Similar statements also hold for a. We now have to the evaluate the $(-h(a)-1)$-th mode of a and apply it to b. The crucial point is that this mode contains at most as many $L_{[-n]}$ as a, see e.g. [26]. $a_{[-h(a)-1]}b$ thus has at most $\lfloor\frac{h(b)+h(a)}{2}\rfloor = \lfloor s-\frac{1}{2}\rfloor = s-1$ $L_{[-n]}$. Since going to the standard basis only decreases their number, it is clear that we cannot obtain $(L_{[-2]})^s\Omega$ from this term.

If we apply the same argument to $d_{[-h(d)+1]}e$, it seems that we could obtain s Virasoro operators. Note however that $d_{[-h(d)+1]}$ annihilates the vacuum and must therefore contain at least one $L_{[-n]}$ with $n \leq 1$. Bringing this operator to the right, commuting through the modes of e, we decrease the number of Virasoro operators at least by one, so that we are again left with at most $s-1$ Virasoro generators.

It therefore follows that the right hand side of (3.10) does not contain the term $(L_{[-2]})^s\Omega$. To satisfy the equality the theory must therefore have a non-trivial null vector. At $c>1$, however, we know that the pure Virasoro theory does not have any non-trivial null vectors. This then leads to the desired contradiction.

4.1 A way out?

This leaves us with the possibility that (3.9) does not imply (3.10), i.e. that f_l and h_j are not holomorphic for $|q|<1$. As we have seen in section 3.3, this can only be the case if the theory has many additional null vectors (that guarantee that all coefficients of the meromorphic functions that would generically appear are actually zero). It is certainly conceivable that this can be achieved with only null-relations at $h>k$,[6] and we do not have any hard argument against this possibility. There is however a curious observation that seems to throw some doubt on this scenario.

As we have explained above, the extremal theory at $k=1$, the Monster theory, has many low-lying null vectors. This property is something one can actually read off from the character. To explain this, let us recall that the partition function of the Monster theory is

$$Z_M(q) = q^{-1} + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + \cdots\ . \tag{4.2}$$

We can read off from this formula that there are $N_1 = 196884$ states at level two; these consist of the stress energy tensor L, as well as the fields W^i we have introduced before.

[6] This is not, though, what happened in the example of the tensor products of the Monster theories: there the null vectors that are responsible for this cancellation appear at or below the level suggested by the order of the differential equation.

k	N_k	M_k	$M_k - N_k^2$
k=1	196884	864299970	-37899009486
k=2	42987520	802981794805760	-1044945080944640
k=3	2593096794	378428749730548169825	371704598747495091389
k=4	81026609428	141229814494885904705260482	141223249183450507046773298

Table 1: The coefficients N_k and M_k for the extremal self-dual theories at $c = 24k$.

Now consider the $N_1^2 = 38\,763\,309\,456$ states

$$L_{-2}L_{-2}\Omega\,, \qquad L_{-2}W^i_{-2}\Omega\,, \qquad W^i_{-2}L_{-2}\Omega\,, \qquad W^i_{-2}W^j_{-2}\Omega\,. \tag{4.3}$$

These states appear at level four. On the other hand, we know from the partition function (4.2) that the total number of states at conformal weight four (above the vacuum) is

$$M_1 = 864\,299\,970 \ll 38\,763\,309\,456 = N_1^2\,. \tag{4.4}$$

Thus it follows from this simple counting argument that there must be many 'null'-relations among the states (4.3); one of them is for example the null vector relation (3.13).

One may ask how this counting argument works for the other extremal self-dual theories. For general k we define N_k and M_k by

$$Z_k(q) = q^{-k} + \cdots + N_k\,q + \cdots + M_k\,q^{k+2}\,, \tag{4.5}$$

where $Z_k(q)$ is the extremal partition function. By the same token as above, the theory will have many null vectors if $M_k - N_k^2 < 0$. For the first few values of k these numbers are listed in table 1.

We have checked these numbers for up to $k = 150$, and the pattern seems to continue — in fact it appears that $N_k^2 \leq d_1\,e^{-d_2 k}\,M_k$ for some constants d_1 and d_2. Thus this counting argument explains why the Monster theory has many low-lying null vectors. It also predicts that the same is true for the theory with $k = 2$, but at least from this point of view, there are no indications that the theories with $k \geq 3$ should have many low-lying null vectors. We regard this as evidence against the possibility that the extremal theories avoid the above contradiction.

5. Conclusions

In this paper we have shown that every modular differential equation of a rational conformal field theory comes from a non-trivial null vector in the Verma module — see (3.9). Generically, the functions f_l and h_j that appear in this identity are analytic in $|q| < 1$, and then (3.9) implies that there is a relation of the form (3.10). At least for the extremal self-dual theories at $c = 24k$ this relation is a non-trivial null relation. This then implies, following the arguments of [11], that these theories are inconsistent for $k \geq 42$.

This analysis is however not completely conclusive since it *is* possible that the functions f_l and h_j appearing in (3.9) are non-holomorphic — indeed, this is what happens for the example of Gaiotto [22] concerning tensor products of the Monster theory (see (3.30)). However, this then requires that the non-holomorphic terms that appear on the right-hand side of (3.9) must actually vanish, thus indicating that there are many other null vector relations (albeit none of the form (3.10)). This is indeed what happens for the case of the tensor product of the Monster theories.

Finally, we have seen from the analysis of the partition functions, that the theories at $k = 1, 2$ must have many non-trivial null vector relations, but that there are no indications (from this point of view) that this should be the case for $k \geq 3$. Taken together we regard this as suggestive evidence for the assertion that the extremal self-dual theories at $c = 24k$ are inconsistent for $k \geq 42$.

The above analysis concerns the extremal bosonic theories at $c = 24k$. It is also interesting to study the supersymmetric generalisations of this set-up; the case with $N = 1$ superconformal symmetry was already analysed in [12], and we have recently (in collaboration with others) studied the case with $N = 2$ superconformal symmetry [27]. In this case the constraints of modular invariance are somewhat stronger since one can not only impose modular invariance of the partition function, but also of the elliptic genus. In fact, using these constraints one can show that the $N = 2$ extremal self-dual theories are inconsistent, except for a few small sporadic values of the central charge [27]. One can also study their modular differential equation, in analogy with what was done in [11]; this will be reported elsewhere.

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A. Vertex operator algebras and Zhu's algebra

The vacuum representation of a (chiral) conformal field theory describes a meromorphic conformal field theory [28]. In mathematics, this structure is usually called a vertex operator algebra (see for example [13, 29] for a more detailed introduction). A vertex operator algebra is a vector space $V = \bigoplus_{n=0}^{\infty} V_n$ of states, graded by the conformal weight. Each element in V of grade h defines a linear map on V via

$$a \mapsto V(a, z) = \sum_{n \in \mathbb{Z}} a_n \, z^{-n-h} \qquad (a_n \in \mathrm{End}\, V) \ . \tag{A.1}$$

In this paper we follow the usual physicists' convention for the numberings of the modes; this differs by a shift by $h - 1$ from the standard mathematical convention that is also, for example, used in [9]. We also use sometimes (as in [9]) the symbol

$$o(a) = a_0 \ . \tag{A.2}$$

Every meromorphic conformal field theory contains an energy-momentum tensor L with modes

$$V(L,z)=\sum_n L_n\, z^{-n-2}\ . \tag{A.3}$$

The modes L_n satisfy the Virasoro algebra.

Since much of our analysis is concerned with torus amplitudes it will be convenient to work with the modes that naturally appear on the torus; they can be obtained via a conformal transformation from the modes on the sphere. More specifically, we define (see section 4.2 of [9])

$$V[a,z]=e^{2\pi i z h_a}\, V(a,e^{2\pi i z}-1)=\sum_n a_{[n]}\, z^{-n-h}\ . \tag{A.4}$$

The explicit relation is then

$$a_{[m]}=(2\pi i)^{-m-h_a}\sum_{j\geq m} c(h_a,j+h-1,m+h-1)\, a_j\ , \tag{A.5}$$

where

$$(\log(1+z))^m(1+z)^{h_a-1}=\sum_{j\geq m} c(h_a,j,m)\, z^j\ . \tag{A.6}$$

This defines a new vertex operator algebra with a new Virasoro tensor whose modes $L_{[n]}$ are given by

$$L_{[n]}=(2\pi i)^{-n}\sum_{j\geq n+1} c(2,j,n+1)L_{j-1}-(2\pi i)^2\frac{c}{24}\delta_{n,-2}\ . \tag{A.7}$$

The appearance of the correction term for $n=-2$ is due to the fact that L is only quasiprimary, rather than primary. Since the two descriptions are related by a conformal transformation to one another, the new modes $S_{[n]}$ satisfy the same commutation relations as the original modes S_n. In particular, the modes $L_{[n]}$ satisfy a Virasoro algebra with the same central charge as the modes L_n.

A.1 Zhu's algebra

One of the key results of Zhu [9] is his characterisation of the highest weight representations of a vertex operator algebra in terms of representations of an associative algebra $A(V)$, usually now called Zhu's algebra. This algebra is defined as the quotient space of V by the subspace $O_{[1,1]}$, where $O_{[1,1]}$ is spanned by elements of the form

$$\oint dz\, \left(V(a,z)\,\frac{(z+1)^{h_a}}{z^2}b\right)\ . \tag{A.8}$$

This definition is motivated by the observation (see for example [30, 31] for a more detailed exposition) that

$$\left\langle \phi_1\middle|\phi_2(1)\oint dz\left(V(a,z)\,\frac{(z+1)^{h_a}}{z^2}b\right)\right\rangle=0 \tag{A.9}$$

provided only that ϕ_1 and ϕ_2 are highest weight states, i.e. are annihilated by all a_n with $n>0$. Thus any combination of two highest weight states defines an element in the dual space of $A(V)$. Zhu showed that also the converse is true; more specifically he proved that $A(V)$ carries the structure of an associative algebra with product

$$a * b = \oint dz \left(V(a,z) \frac{(1+z)^{h_a}}{z} b \right) , \tag{A.10}$$

and that the representations of this associative algebra are in one-to-one correspondence with the highest weight representations of the vertex operator algebra. The product structure (A.10) describes the multiplication of the zero modes on highest weight states; in particular, if ψ is a highest weight state, then

$$o(a)o(b)\,\psi = o(a*b)\,\psi \ . \tag{A.11}$$

For future reference we also note that in Zhu's algebra (see [9], p.296)

$$a*b - b*a = \frac{1}{2\pi i} a_{[-h_a+1]}\, b \ . \tag{A.12}$$

Finally, if V is a rational vertex operator algebra, $A(V)$ is a semisimple algebra.

A.2 The C_2 space

The states of the form (A.8) are not homogeneous with respect to the L_0 grading, even if a and b are. The 'leading term', i.e. the term with the highest conformal weight is the term of the form $a_{-h_a-1}b$. Let us denote the subspace that is generated by states of this form by $O_{[2]}$. (We are using here the same conventions as in [26].) A vertex operator algebra is said to satisfy the C_2 criterion if the quotient space $A_{[2]} = V/O_{[2]}$ is finite dimensional. It is easy to see (and proven in [9]) that the C_2 condition implies that Zhu's algebra is finite dimensional. In fact, the dimension of $A_{[2]}$ provides an upper bound on the dimension of Zhu's algebra. Actually, in many cases these two dimensions agree, but this is not always the case: in particular, the dimension of the C_2 space is always at least two [11], while Zhu's algebra is for example one-dimensional for self-dual theories.

Similarly the C_2 condition also implies that $A_q = V(q)/O_q(V)$ has finite dimension as a $\mathbb{C}[G_4(q), G_6(q)]$-module [9]. To see this, we prove the following lemma that will be useful for the detailed argument in appendix D.

Lemma. *Let ψ_i, $i = 1, \dots N$ be a basis of $A_{[2]}$. Each $v \in V$ can then be written as*

$$v = \sum_{i=1}^{N} \lambda(q)_i \psi_i + \sum_{\kappa} \mu_\kappa H_\kappa(q) \, , \qquad H_\kappa(q) \in O_q(V) \, , \tag{A.13}$$

where the sum over κ is finite and $\lambda_i(q)$ is a polynomial in $\mathbb{C}[G_4(q), G_6(q)]$.

Proof: We note that by construction any $v \in V$ can be written as

$$v = \sum_{i=1}^{N} \tilde{\lambda}_i \psi_i + \sum_{l} r_l \, , \qquad r_l \in O_{[2]} \, , \tag{A.14}$$

where the sum over l is finite. Since each r_l is in $O_{[2]}$, it can serve as the 'head' of an element $H_l(q) \in O_q(V)$, i.e. we can write it as

$$r_l = H_l(q) + \sum_{k\geq 2} G_{2k}(q)\, \hat{r}_{l,k}\,, \tag{A.15}$$

where the states $\hat{r}_{l,k}$ appearing in the 'tail' have a conformal weight which is lower by at least 4. We can thus apply the same procedure again and write $\hat{r}_{l,k}$ as a sum of ψ_i and elements of $O_{[2]}$. Since the conformal weight decreases in each step, this algorithm terminates after a finite number of steps. This shows the Lemma.

Finally we note that since the vertex operator algebras defined by a_n and $a_{[n]}$ are isomorphic, the C_2 condition (formulated for either $a_{-h_a-1}b$ or $a_{[-h_a-1]}b$) implies that the $\mathbb{C}[G_4(q), G_6(q)]$-ideal of $O_q(V)$ in $V[G_4(q), G_6(q)]$ has finite codimension. From this it follows that there is a relation of the type (2.6).

B. Torus recursion relations

In this appendix we briefly sketch the derivation of the recursion relation (2.1); for the detailed argument see [9]. Let us introduce the notation

$$F_{\mathcal{H}}\Big((a^1, z_1), \ldots, (a^n, z_n); q\Big) = z_1^{h_1} \ldots z_n^{h_n}\, \mathrm{Tr}_{\mathcal{H}}\Big(V(a^1, z_1) \cdots V(a^n, z_n)\, q^{L_0}\Big)\ . \tag{B.1}$$

The derivation of (2.1) consists of several steps. We first need the following proposition:

$$\begin{aligned} F_{\mathcal{H}}&\Big((a^1, z_1), (a, w), (a^2, z_2), \ldots, (a^n, z_n); q\Big) \\ &= z_1^{-h_1} \ldots z_n^{-h_n}\, \mathrm{Tr}_{\mathcal{H}}\Big(o(a) V(a^1, z_1) \ldots V(a^n, z_n)\, q^{L_0}\Big) \\ &\quad + \sum_{m\in\mathbb{N}_0} \mathcal{P}_{m+1}\left(\frac{z_1}{w}, q\right) \times F_{\mathcal{H}}\Big((a_{[m-h_a+1]}a^1, z_1), (a^2, z_2), \ldots, (a^n, z_n); q\Big) \\ &\quad + \sum_{j=2}^{n} \sum_{m\in\mathbb{N}_0} \mathcal{P}_{m+1}\left(\frac{z_j}{w}, q\right) \times F_{\mathcal{H}}\Big((a^1, z_1), (a^2, z_2), \ldots, (a_{[m-h_a+1]}a^j, z_j), \ldots, (a^n, z_n); q\Big)\ . \end{aligned} \tag{B.2}$$

Note that there is actually no difference between the terms in the third line and the fourth line — we have only distinguished between them to clarify the derivation below. In fact, it is easy to show that $F_{\mathcal{H}}$ is actually independent of the order in which the (a^j, z^j) (or (a, w)) appear, as must be the case. *Sketch of proof:* The proof is in principle simple: expand out $V(a, w)$ in modes as in (A.1). Commute the zero mode $o(a)$ to the left to get the second line in (B.2); the commutator will eventually be absorbed into the $\mathcal{P}_1(\frac{z_1}{w}, q)$ of the third line, using (C.3). For the other terms in the mode expansion of $V(a, w)$ we commute each mode a_k through the other fields, using

$$[a_k, V(a^j, z_j)] = \sum_{m\in\mathbb{N}} \begin{pmatrix} h+k-1 \\ m \end{pmatrix} V(a_{m-h_a+1}a^j, z_j)\, z_j^{h+k-1-m}\ . \tag{B.3}$$

As a_k is taken past q^{L_0}, we pick up

$$a_k\, q^{L_0} = q^k\, q^{L_0} a_k\ . \tag{B.4}$$

Thus when a_k comes back to its original position, it is multiplied by q^k. We can therefore solve for the original expression to get

$$\begin{aligned} &\mathrm{Tr}_{\mathcal{H}}\Big(V(a^1,z_1)\,a_k\cdots V(a^n,z_n)\,q^{L_0}\Big) \\ &\quad=\frac{1}{1-q^k}\sum_{j=2}^{n}\sum_{l\in\mathbb{N}}\binom{h_a-1+k}{l}z_j^{h_a-1+k-l} \\ &\qquad\qquad\times\mathrm{Tr}_{\mathcal{H}}\Big(V(a^1,z_1)\cdots V(a_{l-h_a+1}a^j,z_j)\cdots V(a^n,z_n)\,q^{L_0}\Big) \\ &\qquad+\frac{q^k}{1-q^k}\sum_{l\in\mathbb{N}}\binom{h_a-1+k}{l}z_1^{h_a-1+k-l}\,\mathrm{Tr}_{\mathcal{H}}\Big(V(a_{l-h_a+1}a^1,z_1)\cdots V(a^n,z_n)\,q^{L_0}\Big)\,. \end{aligned} \tag{B.5}$$

We can then plug this into the original expansion and use the identity

$$\begin{aligned} &\sum_{l\in\mathbb{N}}\sum_{k=1}^{\infty}\left(\binom{h_a-1+k}{l}\frac{1}{1-q^k}x^k+\binom{h_a-1-k}{l}\frac{1}{1-q^{-k}}x^{-k}\right)a_{l-h_a+1}a^j \\ &\qquad\qquad\qquad\qquad=\sum_{m\in\mathbb{N}}\mathcal{P}_{m+1}(x,q)\,a_{[m-h_a+1]}a^j\,, \end{aligned} \tag{B.6}$$

where $\mathcal{P}_{m+1}(x,q)$ is the Weierstrass function, see appendix C. For the terms with $j\neq 1$, $x=z_j/w$, so that we obtain directly the last line of (B.2). For $j=1$, $x=qz_1/w$, and we apply (C.3) to get the third line. Note that for $m=0$ the shift by $2\pi i$ is exactly compensated by the commutator term that comes from the second line. □

We will now use (B.2) to calculate the action of $a_{[-h_a]}$ on one of the inserted operators. We claim that

$$\begin{aligned} &F_{\mathcal{H}}\Big((a_{[-h_a]}a^1,z_1),(a^2,z_2),\ldots,(a^n,z_n);q\Big) \\ &\quad=z_1^{h_1}\ldots z_n^{h_n}\,\mathrm{Tr}_{\mathcal{H}}\Big(o(a)V(a^1,z_1)\cdots V(a^n,z_n)\,q^{L_0}\Big) \\ &\qquad-\pi i\,F_{\mathcal{H}}\Big((a_{[-h_a+1]}a^1,z_1),(a^2,z_2),\ldots,(a^n,z_n);q\Big) \\ &\qquad+\sum_{k=1}^{\infty}G_{2k}(q)\,F_{\mathcal{H}}\Big((a_{[2k-h_a]}a^1,z_1),(a^2,z_2),\ldots,(a^n,z_n);q\Big) \\ &\qquad+\sum_{j=2}^{n}\sum_{m\in\mathbb{N}_0}\mathcal{P}_{m+1}\left(\frac{z_j}{z_1},q\right)F_{\mathcal{H}}\Big((a^1,z_1),\ldots,(a_{[m-h_a+1]}a^j,z_j),\ldots,(a^n,z_n);q\Big)\,. \end{aligned} \tag{B.7}$$

Proof: We can write the first line of (B.7) as

$$\int_C w^{-1}\left(\log\left(\frac{w}{z_1}\right)\right)^{-1}F_{\mathcal{H}}\Big((a,w),(a^1,z_1),\ldots,(a^n,z_n);q\Big)dw\,. \tag{B.8}$$

This can be seen by rewriting $a_{[-h_a]}$ in terms of the original modes, using

$$V(a_l\,a^1,z_1)=\oint dw(w-z_1)^{-l-h_a}\,V(a,w)\,V(a^1,z_1) \tag{B.9}$$

and by the definition of the $c(h,j,m)$,

$$\sum_{j\geq -1} c(h,j,-1)\,(w-z_1)^j\, z_1^{h-1-j} w^{-h} = w^{-1}\left(\log\left(\frac{w}{z_1}\right)\right)^{-1} \,. \tag{B.10}$$

We then use (B.2) to evaluate $F_{\mathcal{H}}$. From (B.8) we see that in the terms that are regular in $w=z_1$, we simply need to replace w by z_1. To evaluate the third line of (B.2) we substitute $z_1 = \exp(2\pi i z_1'), w = \exp(2\pi i w')$, which shows that we obtain the constant term in the w' expansion of $\mathcal{P}_{m+1}(e^{2\pi i w'})$, which can be read off directly from (C.6). $\square$

To get (2.1), we specialise (B.7) to the case $n=1$. Furthermore we use that (see [9])

$$[o(a), V(b,z)] = V(a_{[-h_a+1]}b, z)\,, \tag{B.11}$$

implying that $F_{\mathcal{H}}((a_{[-h_a+1]}b, z); q) = 0$. If we consider the terms of (B.7) of power z^0, we thus obtain

$$\mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[-h_a]}b)\, q^{L_0}\Big) = \mathrm{Tr}_{\mathcal{H}}\Big(o(a)\, o(b)\, q^{L_0}\Big) + \sum_{k=1}^{\infty} G_{2k}(q,y)\, \mathrm{Tr}_{\mathcal{H}}\Big(o(a_{[2k-h_a]}b)\, q^{L_0}\Big)\,. \tag{B.12}$$

B.1 Differential operators

For the determination of the modular differential equation, one of the key steps is the calculation of the differential operators $P_s(D)$, see (2.9). In the following, we give explicit formulae for them for the first few values of s

$$P_1(D) = (2\pi i)^2 D \tag{B.13}$$

$$P_2(D) = (2\pi i)^4 D^2 + \frac{c}{2}\, G_4(q) \tag{B.14}$$

$$P_3(D) = (2\pi i)^6 D^3 + \left(8 + \frac{3c}{2}\right) G_4(q)\,(2\pi i)^2 D + 10c\, G_6(q) \tag{B.15}$$

$$\begin{aligned} P_4(D) = {} & (2\pi i)^8 D^4 + (32+3c)\, G_4(q)\,(2\pi i)^4 D^2 + (160+40c)\, G_6(q)\,(2\pi i)^2 D \\ & + \left(108c + \frac{3}{4}c^2\right) G_4(q)^2 \,. \end{aligned} \tag{B.16}$$

Here c is the central charge of the corresponding conformal field theory.

C. Weierstrass functions and Eisenstein series

Let us define the function

$$\mathcal{P}_k(q_z, q) = \frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\left(\frac{n^{k-1}q_z^n}{1-q^n} + \frac{(-1)^k n^{k-1} q_z^{-n} q^n}{1-q^n}\right)\,, \tag{C.1}$$

which converges for $|q| < |q_z| < 1$. Since $q_z \frac{d}{dq_z}\mathcal{P}_k(q_z,q) = \frac{k}{2\pi i}\mathcal{P}_{k+1}(q_z,q)$, we will concentrate on $\mathcal{P}_1(q_z,q)$. In what follows, we shall be interested in the behaviour around $q_z = 1$. $\mathcal{P}_1^Q(q_z,q,y)$ has a simple pole at $q_z = 1$, but we can find a meromorphic continuation on $|q| < |q_z| < |q|^{-1}$ by rewriting

$$\mathcal{P}_1(q_z,q) = \frac{2\pi i}{1-q_z} - 2\pi i + 2\pi i \sum_{n=1}^{\infty}\left(\frac{q_z^n q^n}{1-q^n} - \frac{q_z^{-n}q^n}{1-q^n}\right)\,. \tag{C.2}$$

A straightforward calculation then shows the identity

$$\mathcal{P}_1(qq_z, q) = \mathcal{P}_1(q_z, q) + 2\pi i \ . \tag{C.3}$$

Introducing the new variable z by $q_z = e^{2\pi i z}$, we want to calculate the Laurent expansion in z around 0. The crucial point is that the coefficients of this Laurent expansion are essentially the Eisenstein series $G_{2k}(q)$ that will eventually appear in (2.1). In fact, expanding q^z in z and using the definition of the Bernoulli numbers,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \,, \tag{C.4}$$

along with the identity

$$B_{2n} = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \zeta(2n) \,, \tag{C.5}$$

we obtain

$$\mathcal{P}_1(q_z, q) = -\frac{1}{z} - \pi i + \sum_{k=1}^{\infty} G_{2k}(q) z^{2k-1} \,, \tag{C.6}$$

where the Eisenstein series are defined by

$$G_{2k}(q) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n} \ . \tag{C.7}$$

The Laurent expansions of the higher $\mathcal{P}_k(q_z, q)$ functions can be directly obtained by

$$\partial_z \mathcal{P}_k(q_z, q) = k \mathcal{P}_{k+1}(q_z, q) \ . \tag{C.8}$$

C.1 The Eisenstein series

The Eisenstein series $G_{2k}(\tau)$ can also be alternatively defined by

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^{2k}} \qquad k \geq 2 \,, \tag{C.9}$$

$$G_2(\tau) = \frac{\pi^2}{3} + \sum_{m \in \mathbb{Z} - \{0\}} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2} \ . \tag{C.10}$$

For $k \geq 2$, $G_{2k}(\tau)$ is a modular form of weight $2k$, i.e.

$$G_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} G_{2k}(\tau) \,, \tag{C.11}$$

whereas $G_2(\tau)$ transforms with a modular anomaly

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - 2\pi i c(c\tau + d) \ . \tag{C.12}$$

We can use G_2 to define a modular covariant derivative: If $f(q)$ is a modular form of weight s, then $D_s f(q)$ is a modular form of weight $s + 2$, where

$$D_s = q\frac{d}{dq} - \frac{s}{4\pi^2} G_2(q) \ . \tag{C.13}$$

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The space of modular covariant functions is given by the ring $\mathbb{C}[G_4(q), G_6(q)]$ that is freely generated by $G_4(q)$ and $G_6(q)$. In particular, all higher $G_{2k}(q)$ can be written as polynomials in $G_4(q), G_6(q)$.

It is also sometimes convenient to work with a different normalisation for the Eisenstein series, so that the constant term is 1; the corresponding series will be noted by $E_n(q)$. For the first few values of n, they are explicitly given as

$$
\begin{aligned}
E_2(q) &= 1 - 24\,q - 72\,q^2 - 96\,q^3 - 168\,q^4 - 144\,q^5 - 288\,q^6 - \cdots\,, \\
E_4(q) &= 1 + 240\,q + 2160\,q^2 + 6720\,q^3 + 17520\,q^4 + 30240\,q^5 + 60480\,q^6 + \cdots\,, \\
E_6(q) &= 1 - 504\,q - 16632\,q^2 - 122976\,q^3 - 532728\,q^4 - 1575504\,q^5 - 4058208q^6 - \cdots\,.
\end{aligned}
$$

The relation between the $G_n(q)$ and $E_n(q)$ is simply $G_n(q) = 2\zeta(n)\,E_n(q)$; for the first few values of n, we have explicitly

$$
G_2(q) = -\frac{(2\pi i)^2}{12}\,E_2(q)\,, \quad G_4(q) = \frac{(2\pi i)^4}{720}\,E_4(q)\,, \quad G_6(q) = -\frac{(2\pi i)^6}{30240}\,E_6(q)\,. \tag{C.14}
$$

Finally, we mention that we have the identities

$$
DE_4 = -\frac{1}{3}E_6\,, \qquad DE_6 = -\frac{1}{2}E_4^2\,, \qquad DE_4^2 = -\frac{2}{3}E_4 E_6\,. \tag{C.15}
$$

D. Radius of convergence

In this appendix we want to explain the details of the calculation leading up to (3.9). We shall also show that if the theory is C_2-finite then the functions $f_l(q)$ and $h_j(q)$ defined in (3.8) have a non-vanishing radius of convergence. We begin by deriving some relations that will be important for the argument in section D.2.

D.1 $A_{[2]}$ relations

In the following we shall assume that the theory is C_2 finite.[7] This means that the space $A_{[2]} = V/O_{[2]}$ is finite-dimensional, say of dimension N. We denote the irreducible representations of the theory by $M_j, j = 1, \ldots, N'$, where $N' \leq N$. Given the close relation between the $A_{[2]}$ space and Zhu's algebra (see appendix A.2) we can then choose a basis $\psi_i,\ i = 1, \ldots, N$ for $A_{[2]}$ such that

$$
\mathrm{Tr}_{M_j}\psi_i = \delta_{i,j}\,, \quad i = 1, \ldots, N'\,, \qquad \mathrm{Tr}_{M_j}\psi_i = 0\,, \quad \forall j\,,\ i = N'+1, \ldots, N\,. \tag{D.1}
$$

For the analysis of section 3.2 it is important to obtain good recursive relations for the vectors that vanish in all traces, i.e. the vectors ψ_i with $i = N'+1, \ldots N$. In a first step we claim that we can write

$$
\psi_i = \sum_l d^{l,i}_{[-h(d^{l,i})+1]} e^{l,i} + \sum_{\kappa \in S_i} \alpha^{\kappa}_{(i)} H_\kappa(0)\,, \qquad i = N'+1, \ldots N\,, \tag{D.2}
$$

[7] If we do not assume C_2 finiteness, the argument can be done essentially the same way, the only difference being that we cannot show that only finitely many correction terms appear. The resulting coefficient functions are then only formal power series in q.

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where each $H_\kappa(q)$ is an element in $O_q(V)$. To prove this we will assume that Zhu's algebra $A(V)$ is semisimple. The proof proceeds in two steps. First we show that if any vector a is trivial in all traces, then a must equal a commutator in Zhu's algebra. This follows for example from a standard theorem of associative algebras, the Wedderburn structure theorem [24]. It states that every semisimple associative algebra is isomorphic to the product of algebras of $n \times n$ matrices over $\mathbb{C}$,

$$A(V) \cong \prod_{i=1}^{N} \mathcal{M}_{n_i}(\mathbb{C}) , \tag{D.3}$$

where n_i is the dimension of the i^{th} irreducible representation M_i of $A(V)$. Assume we are given $a \in A(V)$ such that $\text{Tr}_{M_i}(a) = 0$ for all irreducible representations M_i. By (D.3), a is isomorphic to a blockdiagonal matrix whose blocks all have vanishing trace. It is then a straightforward exercise to show that each such matrix can be written as a sum of commutators. Because of the identity (A.12) we thus find that, up to elements in $O_{[1,1]}$, we have

$$2\pi i \cdot a = 2\pi i \sum_l (d^l * e^l - e^l * d^l) = \sum_l d^l_{[-h(d^l)+1]}\, e^l \ . \tag{D.4}$$

The argument so far implies that up to commutator terms (D.4), ψ_i lies in $O_{[1,1]}$, the subspace by which we quotient to obtain Zhu's algebra $A(V)$. On the other hand, $O_{[1,1]}$ is closely related to $O_q(V)$: for any state in $O_q(V)$,

$$H(q) \equiv a_{[-h_a-1]}b + \sum_{k \geq 2} (2k-1) G_{2k}(q) a_{[2k-h_a-1]}b \, , \tag{D.5}$$

we can formally take the limit $q \to 0$, i.e. we can consider its constant part only. Then we obtain (see [9], Lemma 5.3.2)

$$H(0) = \frac{\pi i}{6} a_{[-h_a+1]}b + 2\pi i \oint dz \left(V(a,z) \frac{(1+z)^{h_a}}{z^2} b \right) \, , \tag{D.6}$$

i.e. up to a commutator term, the limit is in $O_{[1,1]}$. In fact, it is obvious that every element in $O_{[1,1]}$ can be obtained in this manner. Together with (D.4) this then proves the claim (D.2). We should stress that for each i, only finitely many different $d^{l,i}, e^{l,i}$ and H_κ appear.

D.2 Evaluating $K(q)$

As in (3.1) let $K(q) = \sum_r g_r(q) v_r$, $g_r(q) \in \mathbb{C}[G_4(q), G_6(q)]$ be such that

$$\text{Tr}_{M_j} \left(o(K(q))\, q^{L_0 - \frac{c}{24}} \right) = 0 \, , \qquad \forall M_j \ . \tag{D.7}$$

Using the Lemma from appendix A, (A.13), we can write

$$K(q) = \sum_{i=1}^{N} \lambda_i(q) \psi_i + O_q(V) =: K'(q) + O_q(V) \, , \tag{D.8}$$

so that from (D.2)

$$K'(q)=\sum_{i=1}^{N}\lambda_i(q)\psi_i=\sum_{i=1}^{N'}\lambda_i(q)\psi_i+\sum_{i=N'+1}^{N}\lambda_i(q)\sum_l d^{l,i}_{[-h(d^{l,i})+1]}e^{l,i}+\sum_{\kappa\in S}\alpha^\kappa(q)H_\kappa(0)\,, \quad \text{(D.9)}$$

where $\alpha^\kappa(q)=\sum_{i=N'+1}^{N}\lambda_i(q)\alpha^\kappa_{(i)}$ and $S=\bigcup_{i=N'+1}^{N}S_i$ with S_i from (D.2). Now we define

$$N_0(q)=\sum_{i=N'+1}^{N}\lambda_i(q)\sum_l d^{l,i}_{[-h(d^{l,i})+1]}e^{l,i}+\sum_{\kappa\in S}\alpha^\kappa(q)H_\kappa(q) \quad \text{(D.10)}$$

and

$$\Delta_0(q)=K'(q)-N_0(q)=\sum_{i=1}^{N'}\lambda_i(q)\psi_i+\sum_{\kappa\in S}\alpha^\kappa(q)\left(H_\kappa(0)-H_\kappa(q)\right)\ . \quad \text{(D.11)}$$

Since both $K'(q)$ and $N_0(q)$ vanish in all traces for all values of q, also $\Delta_0(q)$ vanishes. Because of our choice of basis (D.1), we know that for $q=0$ each $\lambda_i(0)$ with $i=1,\ldots,N'$ vanishes. It is thus possible to define $\tilde{\Delta}_0(q):=\Delta_0(q)/q$, which is, by construction, still a power series in q.

Next we rewrite the second part of $\tilde{\Delta}_0(q)$ (leaving out the coefficient $\alpha^\kappa(q)$ for the moment) as

$$\begin{aligned} q^{-1}(H_\kappa(0)-H_\kappa(q)) &= \sum q^{-1}(G_{2k}(0)-G_{2k}(q))\,v^\kappa_k=\sum_{i=1}^{N}\lambda^i_\kappa(q)\psi_i+\sum_{\tau\in T}\tilde{\mu}(q)_\kappa{}^\tau H_\tau(q)\\ &=\sum_{i=1}^{N'}\lambda^i_\kappa(q)\psi_i+\sum_{\tau\in T}\tilde{\mu}(q)_\kappa{}^\tau H_\tau(q)\\ &\quad+\sum_{i=N'+1}^{N}\lambda^i_\kappa(q)\sum_l d^{l,i}_{[-h(d^{l,i})+1]}e^{l,i}+\sum_{\tau\in S}\mu(q)_\kappa{}^\tau H_\tau(0)\,, \end{aligned} \quad \text{(D.12)}$$

where in the second equality we have applied the Lemma from appendix A, (A.13), to each v^κ_k, and T is the finite set of all elements of $O_q(V)$ that appear in this process. In the last step we have again used the previous recursion step for the ψ_i with $i=N'+1,\ldots,N$. In particular, the set S is the same as before. Since $\tilde{\Delta}_0(q)$ still vanishes in all traces, we can set $q=0$ to see that the (total) coefficient of each $\psi_i, i=1,\ldots N'$ vanishes. We then define

$$N_1(q)=\sum_\kappa\alpha^\kappa(q)\left(\sum_\tau\tilde{\mu}(q)_\kappa{}^\tau H_\tau(q)+\sum_{i=N'+1}^{N}\lambda^i_\kappa(q)\sum_l d^{l,i}_{[-h(d^{l,i})+1]}e^{l,i}+\sum_{\tau\in S}\mu(q)_\kappa{}^\tau H_\tau(q)\right) \quad \text{(D.13)}$$

and $\Delta_1(q)=\tilde{\Delta}_0(q)-N_1(q)$. It is clear that we can apply the same reasoning to $\Delta_1(q)$ and all the subsequent $\Delta_n(q)$. It is important to note that the only $H_\kappa(q)$ that appear are those with $\kappa\in S$ or $\kappa\in T$. In total we thus obtain the (a priori formal) power series

$$K'(q)=\sum_{n=0}^{\infty}q^nN_n(q)=\sum_l f_l(q)\,d^l_{[-h(d^l)+1]}\,e^l+\sum_{\kappa\in S\cup T}h^\kappa(q)\,H_\kappa(q)\ . \quad \text{(D.14)}$$

To show that it has a non-vanishing radius of convergence, note that for example the last term of $N_n(q)$ is of the form

$$q^n \alpha^\kappa(q)(\underbrace{\mu(q)\cdot\mu(q)\cdots\mu(q)}_{n})_\kappa{}^\tau H_\tau(q) \ . \tag{D.15}$$

By construction, $\mu(q)_\kappa{}^\tau$ is holomorphic for all κ and τ, and thus the $\sup_{|q|<1/2}|\mu(q)_\kappa{}^\tau|$ is finite, so that the norm D of the matrix $(\mu)_\kappa{}^\tau$ is also finite. It thus follows that the radius of convergence ρ is at least $\min\{\frac{1}{D}, 1/2\}$. The other terms in $N_n(q)$ can be dealt with similarly.

This argument therefore shows that the coefficient functions of $H_\kappa(q)$ as well as those of the commutator terms $d^l_{[-h(d^l)+1]}e^l$ have finite radius of convergence.

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